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LETTER TO THE EDITOR

On the use of spacetime transformations in path integration

Antônio B Nassar†‡, J M F Bassalo‡, H S Antunes Neto‡ and Paulo de T Santos Alencar‡

† Department of Physics, University of California at Los Angeles, Los Angeles, CA 90024, USA

‡ Departamento de Física, Universidade Federal do Pará, 66000 Belém-Pará, Brazil

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Abstract. We demonstrate that the use of a spacetime transformation can greatly simplify the evaluation of the propagator for the problem of a particle with time-dependent mass, subject to a time-varying forced harmonic oscillator potential. We show that such a propagator can be easily obtained from the free propagator in the new spacetime coordinate system.

Despite the aesthetic attractiveness of the Feynman path integral formulation of the quantum theory, the evaluation of the propagator for certain time-dependent systems, if done in a straightforward manner, can become an overwhelmingly laborious task (Cheng 1984, 1986). So in order to counter these apparent difficulties spacetime transformation techniques have proved useful in several instances (Nassar *et al* 1986, Junker and Inomata 1985, Dhara and Lawande 1984a, b, Pak and Sokmen 1984). In particular, we have recently demonstrated the usefulness of a certain type of spacetime transformation in deriving the propagator for an explicitly time-dependent quadratic system (Nassar *et al* 1986).

In this letter, we take a further step in our earlier work toward an exact evaluation of the propagator for the problem of a particle with time-dependent mass, subject to a time-varying forced harmonic oscillator potential. Overall, we show that such a propagator can be easily obtained from the free propagator in the new spacetime coordinate system.

Let us start out by writing the Lagrangian of our system as

$$L=\tfrac{1}{2}m(t)\dot{x}^2-\tfrac{1}{2}m(t)\omega^2(t)x^2+xf(t). \tag{1}$$

As thoroughly discussed in our earlier work (Nassar *et al* 1986), the propagator for quadratic systems like (1) can be written as

$$K(x,t;x_0,t_0)=\phi(t,t_0)\exp[(\mathrm{i}/\hbar)S(x,t;x_0,t_0)] \tag{2}$$

where

$$\begin{aligned}S(x,t;x_0,t_0)&=S[\chi(t^*)]\\&=\int_{x_0,t_0}^{x,t}\mathrm{d}t^*\,L[\chi(t^*),\dot{\chi}(t^*),t^*]\\&=\int_{x_0,t_0}^{x,t}\mathrm{d}t^*\,[\tfrac{1}{2}m(t^*)\dot{\chi}^2(t^*)-\tfrac{1}{2}m(t^*)\omega^2(t^*)\chi^2(t^*)+\chi(t^*)f(t^*)]\end{aligned} \tag{3}$$

and

$$\phi(t, t_0) = \phi_0(t_0) \exp\left(-\tfrac{1}{2}\int_{t_0}^{t} \mathrm{d}t^* \frac{1}{m(t^*)} \frac{\partial^2 S}{\partial x^2}\right). \tag{4}$$

The constant $\phi_0(t_0)$ is introduced in (4) in order to match the condition

$$\lim_{t\to t_0} K(x, t; x_0, t_0) = \delta(x - x_0). \tag{5}$$

By performing variation on (3), we obtain

$$\ddot{\chi} + (\dot{m}/m)\dot{\chi} + \omega^2\chi = f/m. \tag{6}$$

Now we make use of the spacetime transformation $\chi(t) \leftrightarrow r(\tau)$ (Leach 1978, Burgan *et al* 1979, Ray 1982):

$$r(\tau) = \chi(t)/C(t) + A(t) \tag{7a}$$

$$\tau = \int^{t} \mu(\lambda)\, \mathrm{d}\lambda. \tag{7b}$$

(The following notation is employed: $\mathfrak{z}(t^* = t) = z$, $\mathfrak{z}(t^* = t_0) = z_0$, $\mathfrak{z}'(t) = \mathrm{d}\mathfrak{z}(t^*)/\mathrm{d}t|_{t^*=t}$ and so on.)

After substitution of (7) into (6), we end up with

$$r'' = 0 \tag{8}$$

provided that

$$2\dot{C}\mu + C\dot{\mu} + C\mu\dot{m}/m = 0 \tag{9a}$$

which leads to

$$\mu = 1/mC^2 \tag{9b}$$

$$\ddot{C} + (\dot{m}/m)\dot{C} + \omega^2 C = 0 \tag{10}$$

and

$$\ddot{A} + \left(\frac{2\dot{C}}{C} + \frac{\dot{m}}{m}\right)\dot{A} + \frac{f}{mC} = 0. \tag{11}$$

The corresponding action and Lagrangian for (8) is

$$\bar{S}(\mathfrak{t}, \tau; \mathfrak{t}_0, \tau_0) = \bar{S}[r(\tau^*)] = \int_{\mathfrak{t}_0,\tau_0}^{\mathfrak{t},\tau} \mathrm{d}\tau^*\, \bar{L}[r'(\tau^*)] \tag{12}$$

where

$$\bar{L} = \tfrac{1}{2}r'^2. \tag{13}$$

Because the respective variations $\delta S = 0$ and $\delta\bar{S} = 0$ are equivalent (i.e. (6) and (8) are equivalent), their corresponding actions may differ by just

$$S = \bar{S} + [g(x, t) - g(x_0, t_0)] \tag{14}$$

where we must have $\delta g = 0$; i.e. g is a function of the initial and end points only. Equation (14) can be rewritten as

$$\int \mathrm{d}t^*\, L = \int \frac{\mathrm{d}t^*}{mC^2}\bar{L} + \int \mathrm{d}t^* \frac{\mathrm{d}g}{\mathrm{d}t^*} \tag{15}$$

implying that

$$L(\chi(t^*), \dot{\chi}(t^*), t) = \frac{\bar{L}(r'(t^*))}{mC^2}\bigg|_{r'(t^*)=(\mathrm{d}/\mathrm{d}t^*)[\chi(t^*)/C(t^*)+A(t^*)]} + \frac{\mathrm{d}g}{\mathrm{d}t^*}. \tag{16}$$

By substituting (1), (13), (7a) and (7b) into (16), we find that

$$\frac{\mathrm{d}g}{\mathrm{d}t^*} = \frac{\mathrm{d}}{\mathrm{d}t^*}\left[\left(\frac{m\dot{C}}{2C}\right)\chi^2 - (mC\dot{A})\chi - \int^{t^*} F(\lambda)\,\mathrm{d}\lambda\right] \tag{17}$$

where $F \equiv m\dot{A}^2C^2/2$, such that we have, from (12)-(14) and (17),

$$S = \frac{1}{2}\frac{(\imath-\imath_0)^2}{(\tau-\tau_0)} + \left[\left(\frac{m\dot{C}x^2}{2C} - \frac{m_0\dot{C}_0x_0^2}{2C_0}\right) - (mC\dot{A}x - m_0C_0\dot{A}_0x_0) - \int_{t_0}^{t} F(\lambda)\,\mathrm{d}\lambda\right]. \tag{18}$$

In turn, by using (4), (7), (9b) and (18), we readily find

$$K = \frac{\phi_0}{\sqrt{C(\tau-\tau_0)}}\,\mathrm{e}^{\mathrm{i}S/\hbar}. \tag{19}$$

The constant factor $\phi_0(t_0)$ can be found by imposing condition (5). It yields

$$\phi_0 = \left(\frac{1}{2\pi\mathrm{i}\hbar C_0}\right)^{1/2} \tag{20}$$

such that the full propagator is

$$K(x, t; x_0, t_0) = \frac{1}{(CC_0)^{1/2}}\exp\left\{\frac{\mathrm{i}}{\hbar}\left[\left(\frac{m\dot{C}x^2}{2C} - \frac{m_0\dot{C}_0x_0^2}{2C_0}\right) - (mC\dot{A}x - m_0C_0\dot{A}_0x_0) - \int_{t_0}^{t} F(\lambda)\,\mathrm{d}\lambda\right]\right\} K_{\mathrm{free}}(\imath, \tau; \imath_0, \tau_0) \tag{21a}$$

where

$$K_{\mathrm{free}}(\imath, \tau; \imath_0, \tau_0) = \left(\frac{1}{2\pi\mathrm{i}\hbar(\tau-\tau_0)}\right)^{1/2}\exp\left(\frac{\mathrm{i}}{2\hbar}\frac{(\imath-\imath_0)^2}{(\tau-\tau_0)}\right). \tag{21b}$$

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